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DEFORMATION MODEL FOR BRITTLE MATERIALS AND THE STRUCTURE OF FAILURE WAVES

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Constitutive equations that describe the experimentally observed failure waves are proposed to model inelastic strains of brittle materials. The complete system of equations is hyperbolic, each equation of this system has divergent form. The model is based on the assumption that continual failure is the process of transition from an intact state to a "fully damaged" state described by the kinetics of the order parameter. The structure of stationary traveling compressive waves is analyzed using a simplified model. It is shown that in a certain range of amplitudes, the wave splits into an elastic precursor and a failure wave.

Key words: *inelastic strain of brittle materials, failure waves, shock-wave structure.*

In the present paper, constitutive equations for modeling inelastic strains of brittle materials are proposed that can be used to describe the so-called failure waves. The fracture of a brittle material under compressive stresses is characterized by the formation of numerous cracks and has a wave nature [1, 2]. Theoretical investigation of the fracture waves is in its infancy [3, 4], and no adequate mathematical model has been proposed for a qualitative analysis and numerical study of the processes mentioned above. In the present paper, constitutive differential equations based on the nonlinear theory of inelastic strains [5] are formulated in the form of a hyperbolic system in which each equation has divergent form. Moreover, the model proposed satisfy the laws of nonequilibrium thermodynamics. Models of this type allow the use of well-developed mathematical and effective numerical methods of solving various problems.

The model proposed is based on the assumption that an element of a material subjected to continual failure undergoes a transition from an intact state to a "fully damaged" state, which can be characterized by elastic moduli different from those of the intact material. This transition is described by an equation for the order parameter with nonlinear kinetics. Moreover, the model takes into account the inelastic deformation of the material that accompanies continual failure. A similar small-strain model and numerical analyses of some problems in good agreement with experimental data [1] were proposed in [3].

In the present paper, the structure of stationary traveling compressive waves is analyzed using a simplified model of continual failure. The investigation technique is similar to the analysis of the shock-wave structure in a medium with relaxation given in [6] and to the method of studying elastoplastic waves in a Maxwell nonlinear medium [5]. In a certain range of amplitudes, the wave splits into an elastic precursor and a failure wave itself, which agrees with the experimentally observed wave structure.

1. Complete System of Constitutive Equations. Following [5], we consider the velocity vector u^i ($i = 1, 2, 3$), the elastic deformation gradient c_{ij} (the elastic distortion tensor [5]), the reference density ρ_0 (the density corresponding to an element of the medium reduced to the state of zero stress field), and the entropy S as the parameters characterizing the state of the medium. As a measure of damage to an element, we introduce the order parameter $\xi \in [0, 1]$.

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In Cartesian coordinates x^i, the complete system of constitutive equations comprises the conservation laws for the momentum, mass, and the elastic distortion tensor, the balance equation for the order parameter, and the energy conservation law:

$$\frac{\partial \rho u^i}{\partial t} + \frac{\partial (\rho u^i u^k - \sigma^{ik})}{\partial x^k} = 0, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial \rho u^k}{\partial x^k} = 0,$$

$$\frac{\partial \rho c^i_j}{\partial t} + \frac{\partial (\rho c^i_j u^k - \rho c^k_j u^i)}{\partial x^k} = -(u^i \beta_j + \varphi^i_j), \qquad \frac{\partial \rho \xi}{\partial t} + \frac{\partial \rho u^k \xi}{\partial x^k} = -\psi, \tag{1}$$

$$\frac{\partial \rho (E + u_i u^i/2)}{\partial t} + \frac{\partial (\rho u(E + u_i u^i/2) - u^i \sigma^k_i)}{\partial x^k} = 0.$$

In addition to the parameters of state of the medium, the following quantities are used in system (1): $\rho = \rho_0 / \det (c^i_j)$ is the density and $\sigma^i_j = -2\rho c^i_n \, \partial E / \partial c^n_j$ is the stress tensor. The specific internal energy E is the closing relation, which should be specified as a function of the parameters of state of the medium (reference density, distortion tensor, order parameter, and entropy):

$$E = E(\rho_0, c^1_1, c^1_2, \ldots, c^3_3, \xi, S).$$

The right sides ψ and $u^i \beta_j + \varphi^i_j$ in the equations for ξ and c^i_j describe the failure kinetics and inelastic strains. The term $u^i \beta_j$ contains the variables β_j in which the equations for elastic distortions are written in divergent form. These auxiliary variables should satisfy the additional laws of conservation [5] implied by system (1):

$$\frac{\partial \rho c^i_j}{\partial x^i} = \beta_j, \qquad \frac{\partial \beta_j}{\partial t} + \frac{\partial (u^i \beta_j + \varphi^i_j)}{\partial x^i} = 0, \tag{2}$$

The variables β_j can be eliminated from system (1), but in this case the equation for the distortion tensor c^i_j loses divergent form:

$$\frac{\partial \rho c^i_j}{\partial t} + \frac{\partial \rho c^i_j u^k}{\partial x^k} - \rho c^k_j \frac{\partial u^i}{\partial x^k} = -\varphi^i_j.$$

An analysis of the equations of the nonlinear Maxwell model of inelastic deformations [5] shows that the characteristics of the system of differential equations are real and, hence, the system is hyperbolic.

It should be noted that the processes governed by Eqs. (1) satisfy the second law of thermodynamics — the law of increasing entropy:

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u^\alpha}{\partial x^\alpha} = Q = \frac{E_{c^i_j} \varphi^i_j + E_\xi \xi}{E_S}.$$

The model constructed is thermodynamically correct provided the production of entropy Q is nonnegative. This condition imposes a constraint on the choice of the functions governing the kinetics of the order parameter and inelastic strains:

$$E_{c^i_j} \varphi^i_j + E_\xi \xi \geqslant 0.$$

Thus, the general equations for modeling the processes of continual failure were formulated above. These equations agree with the laws of thermodynamics, are hyperbolic, and have divergent form. To model the real processes for a specific material, it is necessary to specify the dependence of the internal energy on the parameters of the medium (the equation of state) and the kinetics of inelastic strains and continual failure φ^i_j and ψ, which depend on the parameters of state.

2. Simplified Model of Continual Failure for One-Dimensional Processes and Stationary Traveling Waves. To show the applicability of the model constructed above to the description of continual failure, we formulate simplified one-dimensional equations and, by analyzing stationary traveling waves, verify that for appropriately chosen closing relations, the model provides a qualitative description of the experimentally observed features of the failure waves.

We consider one-dimensional waves that propagate along the axis $x^1 = x$ at a velocity $u^1 = u$ assuming that the other components of the velocity vector vanish: $u^2 = u^3 = 0$. In this case, the stress tensor has diagonal form: $\sigma_{ij} = \sigma_i \delta_{ij}$ (σ_i are the principal stresses). As the strain measure in the one-dimensional case, we use the Hencky logarithmic tensor, which is related to the distortion tensor [5]: $(h_{ij}) = \ln(c_i^k c_j^k)$. In the one-dimensional case, the Hencky tensor also has diagonal form: $h_{ij} = h_i \delta_{ij}$. By virtue of the isotropy of the medium, the stresses and elastic strains in the direction normal to the x-axis are uniform, i.e., $\sigma_2 = \sigma_3$ and $h_2 = h_3$. In this case, the density is related to the Hencky elastic deformation tensor by the formula

$$\rho = \rho_0 \exp(-h_1 - 2h_2), \tag{3}$$

where $\rho_0 = \text{const}$ is the density of the material in the initial undeformed state. Next, we assume that the reference density ρ_0 remains unchanged, i.e., inelastic change in the volume does not occur. Under this assumption, the simplified system of constitutive equations (1) becomes

$$\begin{gathered}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0, \qquad \frac{\partial \rho u}{\partial t} + \frac{\partial(\rho u^2 - \sigma_1)}{\partial x} = 0, \\
\frac{\partial \rho h_2}{\partial t} + \frac{\partial \rho u h_2}{\partial x} = -\varphi, \qquad \frac{\partial \rho \xi}{\partial t} + \frac{\partial \rho u \xi}{\partial x} = -\psi, \\
\frac{\partial \rho(E + u^2/2)}{\partial t} + \frac{\partial(\rho u(E + u^2/2) - u\sigma_1)}{\partial x} = 0.
\end{gathered} \tag{4}$$

In this case, we do not need to introduce the auxiliary variable β_j in accordance with formulas (2) since the one-dimensional equations have divergent form. System (4) implies the entropy balance equation

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u}{\partial x} = Q = \frac{2(E_{h_2} - E_{h_1})\varphi + E_\xi \psi}{E_S}. \tag{5}$$

The low of increasing entropy is satisfied with an appropriate choice of the functions φ and ψ that model the kinetics of inelastic strains and the order parameter.

To close system (4) for one-dimensional wave processes, it is necessary to determine three relations: the equation of state $E = E(\rho_0, c_1^1, c_2^1, \ldots, c_3^3, \xi, S)$ and the functions φ and ψ. The available experimental data are very limited, and additional assumptions are required to determine the closing relations mentioned above. Below, we formulate a simpler model that could be the basis for the further generalization. Ignoring the temperature effects and inelastic strains that accompany continual failure, we obtain an even simpler model for one-dimensional wave processes with the constitutive equations given by

$$\begin{gathered}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0, \qquad \frac{\partial \rho u}{\partial t} + \frac{\partial(\rho u^2 - \sigma_1)}{\partial x} = 0, \\
\frac{\partial \rho h_2}{\partial t} + \frac{\partial \rho u h_2}{\partial x} = 0, \qquad \frac{\partial \rho \xi}{\partial t} + \frac{\partial \rho u \xi}{\partial x} = -\psi.
\end{gathered} \tag{6}$$

System (6) consists of four differential equations in divergent form for four unknowns u, h_1, h_2, and ξ.

The stress tensor (its principal values) is given by

$$\sigma_i = \rho \frac{\partial E}{\partial h_i}, \tag{7}$$

where the internal energy $E = E(h_1, h_2, \xi)$ depends on the Hencky parameters h_1 and h_2 and the order parameter ξ and the density is calculated by formula (3). It should be noted that to calculate stresses by formula (7), one should use the relation for the internal energy $E = E(h_1, h_2, h_3, \xi)$ and set $h_2 = h_3$ only after differentiation.

We next study particular solutions of system (6) in the form of stationary traveling waves. Solutions of this type are determined by the spatial variable $x \in (-\infty, +\infty)$ in an infinite interval, are bounded as $x \to \pm\infty$, and depend only on one variable $L = x - Dt$ ($D = \text{const}$ is the velocity of the stationary traveling wave). We consider the solutions for $D > 0$, i.e., the wave propagation in the x direction. Solutions of this type can be found from the system of ordinary differential equations that follows from (6) if the solution is sought in the form

$$h_1 = h_1(L), \qquad h_2 = h_2(L), \qquad u = u(L), \qquad \xi = \xi(L).$$

Substitution of the above-mentioned functions into system (6) yields the following system of ordinary differential equations for the stationary traveling wave:

$$\frac{d\rho(u-D)}{dL} = 0, \qquad \frac{d\,(\rho u(u-D) - \sigma_1)}{dL} = 0,$$
$$\frac{d\rho(u-D)h_2}{dL} = 0, \qquad \frac{d\rho(u-D)\xi}{dL} = -\psi. \tag{8}$$

The first three equations of system (8) can be integrated and reduced to three algebraic equations — relations at the discontinuity. As a result, the system becomes

$$[\rho(u-D)] = 0, \qquad [\rho u(u-D) - \sigma_1] = 0,$$
$$[\rho(u-D)h_2] = 0, \qquad \frac{d\rho(u-D)\xi}{dL} = -\psi. \tag{9}$$

Here $[F] = F - F_0$ is the discontinuity in F and F_0 and F are the values ahead of the wave ($L = +\infty$) and behind it ($L = -\infty$), respectively. The last ordinary differential equation of system (9) determines the wave structure in the interval $-\infty < x < +\infty$.

3. Choice of Closing Relations. To study the failure waves in specific materials, we use the closing relations for system (6): the right side in the equation for ξ, which describes the kinetics of the order parameter, and the equation of state (the specific internal energy E).

We first determine the function ψ subject to the constraints that the function should: 1) ensure positive entropy production in Eq. (5); 2) vanish for $\xi = 0$ and $\xi = 1$. The function $\psi = \psi_0(1-\xi)E_\xi$ satisfies the condition of nonnegative entropy production as $\psi E_\xi \geqslant 0$ and vanishes for $\xi = 1$. Under the assumption that $E_\xi = 0$ for an intact stress-free material (below, this assumption is shown to be valid), the function ψ chosen in this manner satisfies the necessary requirements. We note that ψ_0 can depend on the parameters of state of the material (stresses and ξ).

We choose the equation of state based on the above statement that the process of continual failure is modeled by transition of the initial intact phase ($\xi = 0$) to a "fully damaged" phase ($\xi = 1$). In an intermediate state, an element of the medium can be treated as a mixture of components of the intact and "fully damaged" materials. According to [3], the order parameter ξ can be identified with the volume concentration of the "fully damaged" material in the mixture.

In [3], a procedure is proposed to derive the equation of state of the mixture for known equations of state of the intact and "fully damaged" materials. This procedure is based on the hypothesis that the specific internal energy of the mixture E is the average over the mass concentration of the specific internal energies E_1 and E_2 of the intact and "fully damaged" materials, respectively:

$$\rho E = \xi\rho_2 E_2 + (1-\xi)\rho_1 E_1$$

[$\rho = \xi\rho_2 + (1-\xi)\rho_1$ is the density of the mixture]. To express the parameters of state of the mixture in terms of the parameters of state of the components, we use additional hypotheses. Assuming that the stress field in an element of the mixture is in equilibrium (i.e., the stresses in the elastic intact and "fully damaged" components are equal) [3] and that the elastic strain of the mixture is the average of the volume concentration of the elastic strains of the intact and fully damaged materials, one can obtain a closed-form dependence of the internal energy on the elastic strain of the mixture and the order parameter. In the case where the internal energy of the intact and "fully damaged" materials is described by Hooke's law, the averaging procedure leads to the following equation of state for the mixture [3]:

$$E(h_1, h_2, h_3, \xi) = \frac{\lambda(\xi)}{2\rho_0}\,(h_1 + h_2 + h_3)^2 + \frac{\mu(\xi)}{\rho_0}\,(h_1^2 + h_2^2 + h_3^2). \tag{10}$$

The moduli of the damaged material are given by the formulas

$$\lambda = \frac{K_1 K_2}{\tilde{K}} - \frac{2}{3}\,\frac{\mu_1\mu_2}{\tilde{\mu}}, \qquad \mu = \frac{\mu_1\mu_2}{\tilde{\mu}}, \tag{11}$$

$$\tilde{K} = \xi K_1 + (1-\xi)K_2, \qquad \tilde{\mu} = \xi\mu_1 + (1-\xi)\mu_2, \qquad K_2 = \lambda_2 + (2/3)\mu_2, \qquad K_1 = \lambda_1 + (2/3)\mu_1.$$

We have $\lambda = \lambda_1$ and $\mu = \mu_1$ for $\xi = 0$ (intact material) and $\lambda = \lambda_2$ and $\mu = \mu_2$ for $\xi = 1$ (fully damaged material).

Below, we show that the equation of state (10) can be used for a qualitative description of failure-wave propagation.

4. Equations for Describing the Structure of a Failure Wave Propagating in a Stress-Free Material. We construct a solution of system (9) that describes the structure of a failure wave propagating in a stress-free material. Since solution of (9) is bounded as $L = \pm\infty$, the derivatives of all unknown functions with respect to the variable L vanish as $L = \pm\infty$. Let u_0, ρ_0, h_2^0, and ξ_0 be known parameters of state ahead of the wave ($\xi = +\infty$). Then, system (9) becomes

$$\rho(u-D) = \rho_0(u_0-D) = m, \qquad \rho u(u-D) - \sigma_1 = \rho_0 u_0(u_0-D) - \sigma_1^0,$$

$$\rho(u-D)h_2 = \rho_0(u_0-D)h_2^0, \qquad \frac{d\rho(u-D)\xi}{dL} = -\psi,$$

where m is the mass flux through the wave.

We study the wave propagating in a stress-free immovable material ($u_0 = 0$) and intact material ($\xi_0 = 0$), in which the elastic strain in the direction normal to the x axis vanishes ($h_2^0 = 0$). In this case, the third equation of this system implies that $h_2 = 0$. Then, the strain along the x axis can be written as a function of density: $h_1 = \ln \rho_0/\rho$.

Using the definition of the mass flux through the wave m and introducing the notation $U = u - D$, we finally obtain the system for studying the structure of the stationary traveling wave:

$$\rho U = \rho_0 U_0 = m, \qquad \frac{m^2}{\rho} - \sigma_1(\rho,\xi) = \frac{m^2}{\rho_0} - \sigma_1^0, \qquad m\,\frac{d\xi}{dL} = -\psi. \tag{12}$$

System (12) contains three equations for the four unknowns ξ, ρ, U, and m. To solve this system, it is necessary to specify the quantity m or another parameter of state behind the wave (as $L = -\infty$). This system, however, is not always solvable; its solvability conditions are given below.

The problem of the structure of the failure wave is solved in two stages. We first find the parameters of state behind the wave and then construct a solution that joins the states ahead of the wave ($L = +\infty$) and behind it.

We assume that the stress behind the wave ($L = -\infty$) is known: $\sigma_1 = -P$. Provided the solution is bounded as $L \to -\infty$, we find the other parameters of state behind the wave. The boundedness condition implies that $d\xi/dL = 0$ for $L = -\infty$. As a result, we obtain the system of algebraic equations for determining the mass flux through the wave m, the density ρ, and the order parameter ξ behind the wave:

$$m^2/\rho + P = m^2/\rho_0 - \sigma_1^0, \qquad \psi = \chi E_\xi = 0, \qquad \sigma_1(\rho,\xi) = -P.$$

Assuming that the right side ψ in the equation for ξ vanishes for $\xi = 1$, we arrive at the system for determining m and ρ behind the wave:

$$m^2/\rho + P = m^2/\rho_0 - \sigma_1^0, \qquad \sigma_1(\rho,1) = -P. \tag{13}$$

Provided the equation of state is convex, we obtain unique values of m and ρ for each specified value of P. It is therefore possible to plot a curve $P(V)$ in the plane (P,V) ($V = 1/\rho$ is the specific volume), which we call the Hugoniot adiabat of the "fully damaged" material. We note that solution of system (13) exists for $m^2 > (\partial\sigma_1/\partial V)\Big|_{\rho=\rho_0,\,\xi=1}$. From this condition, it follows that the failure-wave propagation velocity in the fully failed material is higher than the sound velocity.

We determine the elastic shock and Hugoniot adiabat of the elastic material. Since the wave propagates over the stress-free intact material ($\xi = 0$), the system admits a discontinuous solution for which $\xi = 0$ everywhere for $L \in (-\infty, +\infty)$ and the remaining parameters are determined from the conditions at the discontinuity

$$\rho U = \rho_0 U_0 = m, \qquad m^2/\rho - \sigma_1(\rho,0) = m^2/\rho - \sigma_1^0.$$

Assuming that the stress $\sigma_1 = -P$ is specified behind the wave (discontinuity), for each value of P we find the mass flux through the discontinuity m and the density behind the discontinuity ρ from the system

$$m^2/\rho + P = m^2/\rho - \sigma_1^0, \qquad \sigma_1(\rho,0) = -P. \tag{14}$$

Then, we construct the curve $P(V)$, which we call the Hugoniot adiabat of the elastic material. The elastic shock as the solution of system (13) exists for $m^2 > (\partial\sigma_1/\partial V)\Big|_{\rho=\rho_0,\,\xi=0}$. From this condition, it follows that the failure-wave propagation velocity in the elastic intact material is higher than the sound velocity.

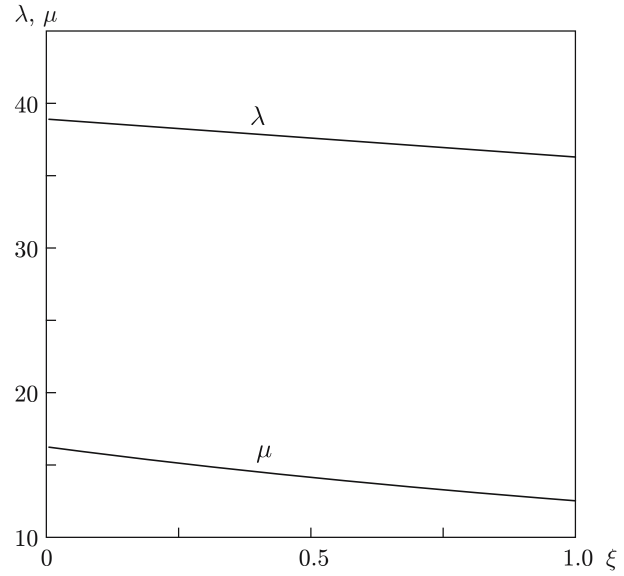

Fig. 1

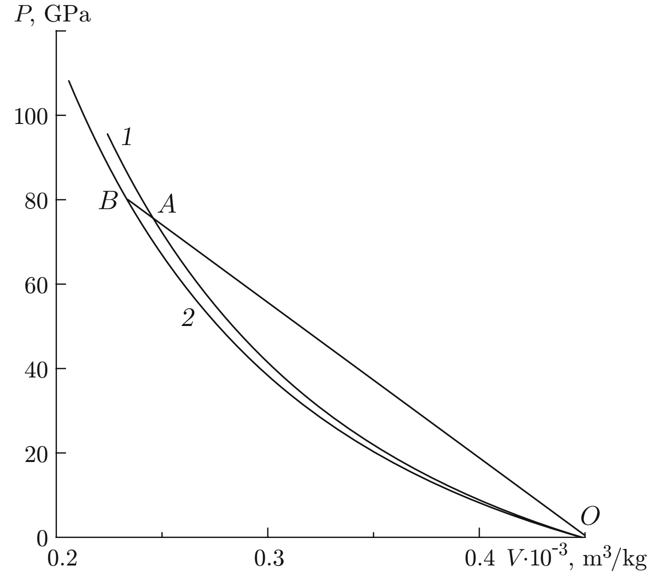

Fig. 2

Fig. 1. Moduli λ and μ versus the order parameter ξ.

Fig. 2. Hugoniot adiabats for elastic intact material (1) and "fully damaged" material (2): point A refers to the elastic shock and segment AB refers to the continuous failure wave.

Continuous portions of the solution of the wave structure are obtained by integrating the system

$$\frac{dL}{d\xi} = -\frac{m}{\psi}, \qquad \frac{m^2}{\rho} - \sigma_1(\rho, \xi) = \frac{m^2}{\rho_0} - \sigma_1^0. \tag{15}$$

Since the order parameter ξ varies in the range from 0 to 1, the coordinate L can be found using system (15).

5. Example of Solving the Problem of the Structure of the Failure Wave. As the material, we choose borosilicate glass (PYREX) with the following parameters of the intact (fully damaged) materials: $\lambda_1 = 38.9$ GPa ($\lambda_2 = 36.3$ GPa), $\mu_1 = 16.2$ GPa ($\mu_2 = 12.5$ GPa), and $\rho_1 = \rho_2 = 2230$ kg/m^3.

Since the set of constants of the intact material given in [1] contains only the density and propagation velocity of the longitudinal sound waves, we choose λ_1 and μ_1 in such a manner that the value of $(\lambda_1 + 2\mu_1)/\rho_1$ is equal to the squared sound velocity of the longitudinal waves in the intact material (5560 m/sec). We further assume that the density of the material remains unchanged in the stress-free state. The choice of the sound velocities in the "fully damaged" material is of hypothetical character and motivated by the necessity of obtaining closer agreement with experimental data on shock-wave loading.

Figure 1 shows the moduli λ and μ versus the order parameter ξ. One can see that dependence (11) is nonlinear, but it is almost linear for the chosen values of the moduli of the damaged material.

Figure 2 shows the Hugoniot curves for the intact and "fully damaged" materials for the waves propagating in the stress-free immovable material ($\rho_0 = 2230$ kg/m^3 and $u_0 = 0$ ahead of the wave). One can see that the Hugoniot curve for the damaged material lies below that for the intact material. Thus, each "typical" failure wave can be related to the segment OAB which intersects the Hugoniot curves at the points corresponding to the elastic shock (point A) and the continuous failure wave (segment AB). In addition to the solutions composed of the elastic shock accompanied by the wave of transition from the intact to failed material, there can exist continuous solutions in the form of only the transition wave. Waves of this type exist for $\partial P_2/\partial V < m^2 < \partial P_1/\partial V$ (P_2 and P_1 are the Hugoniot curves for the fully failed and intact materials, respectively).

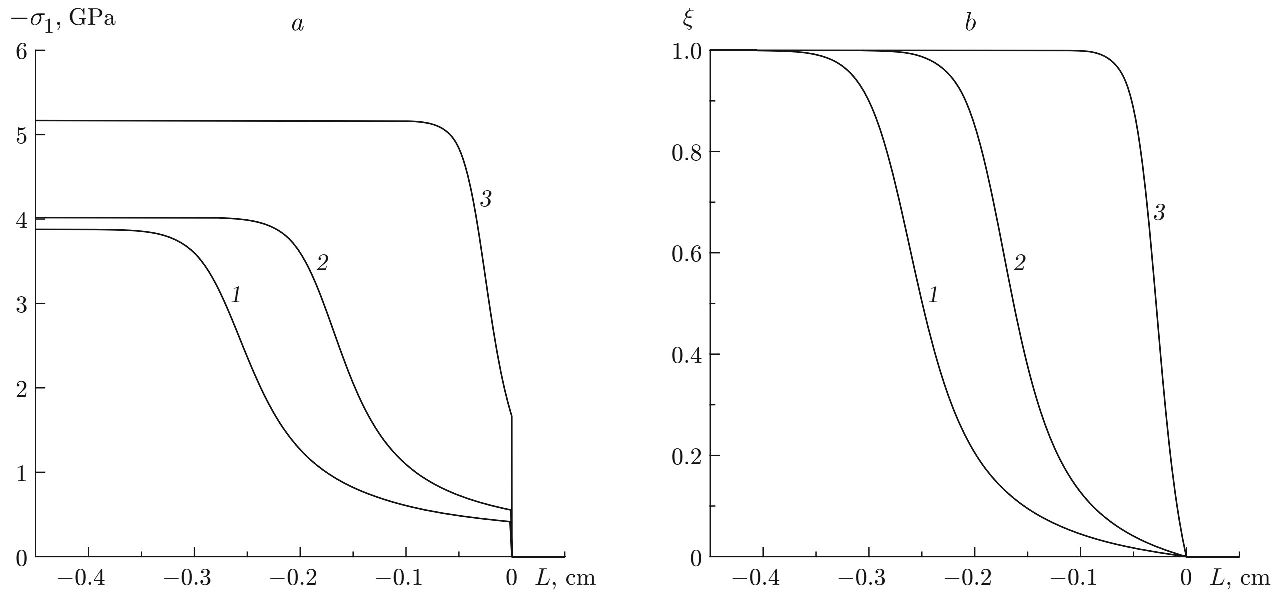

Fig. 3. Distributions of the stresses σ_1 (a) and the order parameter ξ (b) for various failure waves: curve 1 refers to $D = 5585$ m/sec, $u = 311$ m/sec, $\sigma_1 = 3.88$ GPa, and $\rho = 2362$ kg/m^3, curve 2 refers to $D = 5593$ m/sec, $u = 322$ m/sec, $\sigma_1 = 5.16$ GPa, and $\rho = 2366$ kg/m^3, and curve 3 refers to $D = 5659$ m/sec, $u = 409$ m/sec, $\sigma_1 = 5.16$ GPa, and $\rho = 2404$ kg/m^3.

We give the solution of the problem of the wave structure for a certain function ψ describing the kinetics of the order parameter, which was chosen such that the predicted wave parameters were close to the experimental data of [1]. The solution constructed is based on the kinetics

$$\frac{\partial \xi}{\partial t} + u \frac{\partial \xi}{\partial x} = -\frac{1}{\rho}\,\psi,$$

where $\psi = (1-\xi)\psi_0 E_\xi$, $\psi_0 = K\rho(|\sigma|/\sigma_0)^\alpha$, $|\sigma| = |\sigma_1 - \sigma_2|$ for the one-dimensional strains considered, $\sigma_0 = 0.45$ GPa, $K = 2 \cdot 10^4$ sec/m^2, and $\alpha = 0.5$.

Figure 3a shows the distribution of the stress σ_1 for three different failure waves propagating in an immovable stress-free elastic material ($u_0 = 0$, $\rho = \rho_0$, and $\xi_0 = 0$ ahead of the wave). One can see that in all three cases the wave consists of an elastic precursor followed by a smooth failure wave. The characteristic thickness of the transition zone from the intact to fully damaged state is fractions of a centimeter, and it decreases with increasing amplitude. If the wave amplitude is large enough, the elastic precursor and transition zone merge and the failure wave looks like one wave. The order-parameter profiles for these waves are shown in Fig. 3b. One can see that $0 \leqslant \xi \leqslant 1$ and the rate of variation of ξ increases with the wave amplitude. We note that curve 3 is close to the experimental curve of [1]. However, the model considered above ignores the inelastic strain occurring in the process of brittle failure and the plastic strain of the fully failed material. Accounting for these factors will provide a more accurate description of experimental data.

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